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Results in Mathematics

Sequence space representations of spaces of Whitney functions

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Abstract. Let K be a sequence of closed intervals in $\mathbb{R}$ tending to a point. We consider the isomorphic classification problem of Köthe spaces representing the spaces $\mathcal{E}(K)$ of Whitney functions defined on K and their subspaces $\mathcal{E}_0(K)$ of functions on K vanishing at the point of accumulation of intervals. As a tool we use linear topological invariants.

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I. Introduction. The isomorphic classification of the spaces of infinitely differentiable Whitney functions on compact sets $K \subset \mathbb{R}$ has been considered in [1], [2], [3] and [4]. The problem of existence of basis in spaces $\mathcal{E}(K)$ is still open in general. Excepting the classical result of Mitiagin [5] with $K = [-1,1]$ the basis in the space $\mathcal{E}(K), K \subset \mathbb{R}$ has only been constructed in [6] (see also [7]) for K being a sequence of closed intervals tending to a point. In this case, since the spaces of infinitely differentiable Whitney functions are nuclear Fréchet spaces, we can represent the space $\mathcal{E}(K)$ as a Köthe space of coefficients of basis expansions. In the present paper we deal with the Köthe space representations of the space $\mathcal{E}(K)$ and its subspace $\mathcal{E}_0(K)$ of Whitney functions on K vanishing at the point of accumulation of intervals.

We consider a sequence of disjoint closed intervals $I_k = [a_k, b_k]$ such that $(a_k), (b_k)$ monotonically decrease to 0. Let $d_k = b_k - a_k$ denote the length of I_k and $h_k = a_k - b_{k+1}$ be the distance between I_k and I_{k+1}. Suppose $d_k \searrow 0, h_k \searrow 0$ and $d_k \le h_k < 1$ for each k. Then for the compact set $K = \{0\} \cup \bigcup_{k=1}^{\infty} I_k$ we denote by $\mathcal{E}_0(K)$ the subspace of $\mathcal{E}(K)$ consisting of functions which are equal to 0 with all their derivatives at the point 0.

It was shown in [7] under additional assumption

$$\exists M_0 : h_k \ge b_k^{M_0}, \quad k \in \mathbb{N} \tag{1}$$

that the space $\mathcal{E}_0(K)$ has a basis $\{e_{n,k}\}_{n,k \in \mathbb{N}}$ where $\mathbb{N} = \{1, 2, 3 \cdots\}$. The corresponding Köthe space $K(A')$ is given by the matrix $A' = (a'_{nkp})$, where $a'_{nkp} = n^{2u_p} d_k^{-u_p} h_k^{u_p - p}, \quad u_p = \min(n, p), \quad p \in \mathbb{Z}^+ = \{0, 1, \cdots\}$. It is clear that if $A = A_{(d,h)} = (a_{nkp})$, where

$$a_{nkp} = n^p d_k^{-u_p} h_k^{-p}, \ \ u_p = \min(n, p), \tag{2}$$

then $K(A')$ is isomorphic to $K(A)$. By [6], the analogous matrix represents the basis in the space $\mathcal{E}(K)$, but here the parameters h_k have a slightly different geometric meaning. Thus the analysis of linear topological structure of the Köthe spaces $K(A_{(d,h)})$ should be of interest for the isomorphic classification of the spaces of Whitney functions.

II. Property $\mathcal{D}_\varphi$. Now we consider a linear topological invariant defined by interpolation property $\mathcal{D}_\varphi$. This property was considered by Vogt [8] and Tidten [9] (and called DN_φ by them) and by Goncharov and Zahariuta [10], [11] and [12].

Definition 1. Let φ be a continuous, increasing function such that $\varphi(t) \geq t$ for $t > 0$. A Fréchet space X with a fundemental sequence of seminorms $(\|\cdot\|_p)$ is said to have the property $\mathcal{D}_\varphi$ if $\exists p, \forall q, \exists r, m, C$ such that

$$\|x\|_q \leq \varphi^m(t)\|x\|_p + \frac{C}{t}\|x\|_r, \quad t > 0, x \in X.$$

Proposition 1. (see e.g. [12] or [2]) A Köthe space $K(b_{ip})$ has the property $\mathcal{D}_\varphi$ if and only if

$$\exists p \; \forall q \; \exists r, m, C : \forall i \quad \frac{b_{ip}}{b_{iq}} \, \varphi^m \left(C \frac{b_{ir}}{b_{iq}} \right) \geq 1. \tag{3}$$

The following proposition (compare this with Theorem 2 in [7]) is an immediate application of Proposition 1.

Proposition 2. Let $A_{(d,h)}$ be a matrix as in (2). The space $K(A_{(d,h)})$ has the property $\mathcal{D}_\varphi$ if and only if

$$\exists N, C : d_k \geq \varphi^{-N}(C h_k^{-N}), \quad \forall k. \tag{4}$$

III. Zahariuta Compound Invariants. Linear topological invariants (such as approximative and diametral dimensions) have been used for isomorphic classification of non-normed linear topological spaces by Pełczyński [13], Kolmogorov [14], Bessaga, Pełczyński and Rolewicz [15], Mitiagin [5] et al. In this work we consider linear topological invariants introduced by Zahariuta [16], [17], [18]. See also [19] for an extensive consideration of these invariants.

Now let us recall the definition of the following characteristic

$$\beta(V, U) = \sup_L \{\dim L : L \cap U \subset V\}$$

where U, V are absolutely convex, absorbent sets in a locally convex space X and supremum is taken over all finite dimensional subspaces L of X. It is clear that if $T : X \to Y$ is an isomorphism and $T(V) \subset \tilde{V}$ and $\tilde{U} \subset T(U)$ then $\beta(T(V), T(U)) \leq \beta(\tilde{V}, \tilde{U})$

We will consider Zahariuta compound invariants of the form

$$\beta(W_1 \cap W_2, \overline{conv}(W_3 \cup W_4)) \tag{5}$$

for various choices of neighborhoods W_1, W_2, W_3, W_4.

Let X be a Köthe space and A be the set of all sequences with positive terms. For $(a_i) \in A$, let $B(a_i) = \{x \in X : \sum_{i=1}^{\infty} |x_i| a_i \leq 1\}$.

Proposition 3. (See e.g. [20], Proposition 3) If $(a_i), (b_i)$ are in the set A then $\beta(B(b_i), B(a_i)) = |\{i : b_i \leq a_i\}|$.

Here and later $|Z|$ denotes the cardinality of the set $Z \subset \mathbb{N}$ if Z is a finite set and $+\infty$ if Z is an infinite set.

The following proposition can be easily proved.

Proposition 4. Let $c_i = \min\{a_i, b_i\}$, $d_i = \max\{a_i, b_i\}$. Then

$$B(d_i) \subset B(a_i) \cap B(b_i) \subset 2B(d_i), \quad \overline{conv}(B(a_i) \cup B(b_i)) = B(c_i).$$

IV. Asymptotics of the function $\beta(\tau U_p \cap tU_r, U_q)$.

Now we fix $p, q, r \in \mathbb{Z}_+$ such that $p < q < r$ and $t, \tau \in \mathbb{R}_+$ where t will be large and τ will be near 0. We choose the neighbourhoods in (5) as

$$W_1 = \tau U_p = B\left(\frac{1}{\tau} b_{ip}\right), \; W_2 = tU_r = B\left(\frac{1}{t} b_{ir}\right), \; W_3 = W_4 = U_q = B(b_{iq}).$$

Then, using Proposition 4, we get (see also [17])

Proposition 5.

$$\left|\left\{i : \frac{b_{iq}}{b_{ip}} \geq \frac{1}{\tau}, \frac{b_{ir}}{b_{iq}} \leq t\right\}\right| \leq \beta(\tau U_p \cap tU_r, U_q) \leq \left|\left\{i : \frac{b_{iq}}{b_{ip}} \geq \frac{1}{2\tau}, \frac{b_{ir}}{b_{iq}} \leq 2t\right\}\right|.$$

Now we are ready to estimate the function β for the space $K(A)$ given in (2).

We first obtain a lower bound for β. To do this, take $n = q$. Then

$$\frac{a_{nkq}}{a_{nkp}} = \left(\frac{q}{d_k h_k}\right)^{q-p}, \quad \frac{a_{nkr}}{a_{nkq}} = \left(\frac{q}{h_k}\right)^{r-q}.$$

Proposition 5 now gives

$$\beta \geq \left|\left\{k : \left(\frac{d_k h_k}{q}\right)^{q-p} \leq \tau, \; \left(\frac{h_k}{q}\right)^{r-q} \geq \frac{1}{t}\right\}\right| = [k_2 - k_1]^+,$$

where

$$k_1 = \min\left\{k : d_k h_k \leq q\tau^{\frac{1}{q-p}}\right\}, \; k_2 = \max\left\{k : h_k \geq q \, \frac{1}{t^{\frac{1}{r-q}}}\right\} \tag{6}$$

and $[m]^+$ denotes m for $m > 0$, and 0 otherwise.

In order to get an upper bound for β, we put the following restriction on parameters;

$$2\tau < \left(\frac{1}{2t}\right)^{q-p}. \tag{7}$$

Let $Z_n := \{k \in \mathbb{N} : \frac{a_{nkq}}{a_{nkp}} \geq \frac{1}{2\tau}, \frac{a_{nkr}}{a_{nkq}} \leq 2t\}$. We will show that $Z_n \neq \emptyset$ only in the case $p < n \leq q$.

Indeed; let us first examine the case $n \leq p$. Here, $u_q - u_p = u_r - u_q = 0$ and $Z_n = \{k \in \mathbb{N} : (\frac{n}{h_k})^{q-p} \geq \frac{1}{2\tau}, (\frac{n}{h_k})^{r-q} \leq 2t\}$. Applying (7), we get $\frac{h_k}{n} < (\frac{h_k}{n})^{r-q}$, which is impossible.

Under the assumption $q < n \leq r$, we have $Z_n = \{k \in \mathbb{N} : (\frac{n}{d_k h_k})^{q-p} \geq \frac{1}{2\tau}, (\frac{n}{h_k})^{r-q} d_k^{q-n} \leq 2t\}$. If there exists a pair (n, k) satisfying these inequalities, then (7) implies

$$\left(\frac{d_k h_k}{n}\right)^{q-p} < \left(\frac{1}{2t}\right)^{q-p}.$$

Therefore, $\frac{d_k h_k}{n} < (\frac{h_k}{n})^{r-q} d_k^{n-q}$, and $1 < h_k^{r-q-1}$. But this contradicts our assumption $h_k < 1$.

Now, suppose $n > r$. Then, $Z_n = \{k : (\frac{n}{d_k h_k})^{q-p} \geq \frac{1}{2\tau}, (\frac{n}{d_k h_k})^{r-q} \leq 2t\}$. Using (7), we again obtain a contradiction. Thus, $\beta \leq |\bigcup_{n=p+1}^{q} Z_n|$, where $Z_n \subset \left\{k \in \mathbb{N} : (d_k h_k)^{q-p} \leq (2q)^{q-p}\tau, h_k^{r-q} \geq \frac{1}{2t}\right\}$.

We conclude finally that $\beta \leq (q-p)[k_4 - k_3]^+$, where

$$k_3 = \min\{k : d_k h_k \leq 2q\tau^{\frac{1}{q-p}}\},\ k_4 = \max\left\{k : h_k \geq \left(\frac{1}{2t}\right)^{\frac{1}{r-q}}\right\}. \tag{8}$$

Combining the estimates of the function β we get the first main result of this note.

Theorem 1. Let $X = K(A_{(d,h)})$. Then under the assumption (7),

$$[k_2 - k_1]^+ \leq \beta(\tau U_p \cap t U_r, U_q) \leq (q-p)[k_4 - k_3]^+,$$

where $k_i, i = 1, 2, 3, 4$ are defined in (6), (8).

V. Example of non-isomorphic spaces $K(A_{(d,h)})$ not distinguishable by $\mathcal{D}_\varphi$.

Fix $\lambda > 1$. Let $b_k = \exp(-(\ln k)^\lambda)$, $d_k = \exp(-\exp(\ln k)^\lambda)$, $a_k = b_k - d_k$, $I_k = [a_k, b_k]$ for $k \in \mathbb{N}$. We consider the Köthe space $X_\lambda = K(A_{(d,h)})$ corresponding to the space $\mathcal{E}_0(K_\lambda)$ with $K_\lambda = \{0\} \cup \bigcup_{k=1}^{\infty} I_k$. Since $(\ln(k+1))^\lambda \sim (\ln k)^\lambda + \frac{\lambda(\ln k)^{\lambda-1}}{k}$ asymptotically, $h_k = a_k - b_{k+1} \sim b_k \frac{\lambda(\ln k)^{\lambda-1}}{k}$ as $k \to \infty$. Therefore, $h_k \geq b_k^2$ for $k \geq k_0$. This gives (1) and the isomorphism

$\mathcal{E}_0(K_\lambda) \cong X_\lambda$. To deal with the function β, we can use here the following estimates

$$\begin{array}{ll} k_1 \le \min\{k : d_k \le \tau\} & , \quad k_2 \ge \max\{k : b_k \ge (\frac{1}{t})^{\frac{1}{2r}}\}, \\ k_3 \ge \min\{k : d_k \le \tau^{\frac{1}{q}}\} & , \quad k_4 \le \max\{k : b_k \ge t^{-2}\} \end{array} \tag{9}$$

which are fulfilled asymptotically when $t \to \infty, \tau \to 0$.

The proof of these estimates is straightforward.

Let us take $\tau = \exp(-\ln^2 t)$, which clearly satisfies (7) for large enough t. Then (9) implies

$$k_1 \le \exp(2 \ln \ln t)^{1/\lambda}, k_2 \ge \exp(\frac{1}{2r} \ln t)^{1/\lambda}, k_4 \le \exp(2 \ln t)^{1/\lambda}. \tag{10}$$

Therefore, for distinct values of parameter λ, we have the counting functions with different asymptotics.

Proposition 6. If $\lambda \neq \nu$, then the spaces X_λ, X_ν are not isomorphic.

Proof. Suppose contrary to our claim, that $X_\lambda \cong X_\nu$ with $1 < \nu < \lambda$. Then we have $\forall p_1 \, \exists p \, \forall q \, \exists q_1 \, \forall r_1 \, \exists r \, \exists C$ such that

$$\beta(\tau U_p^{(\nu)} \cap t U_r^{(\nu)}, U_q^{(\nu)}) \le \beta(C\tau U_{p_1}^{(\lambda)} \cap Ct U_{r_1}^{(\lambda)}, U_{q_1}^{(\lambda)})$$

where $(U_p^{(\nu)})$ and $(U_p^{(\lambda)})$ denote the neighborhood bases in X_ν and X_λ respectively. By using Theorem 1 and (10), we get

$$\exp\left(\frac{1}{2r} \ln t\right)^{1/\nu} - \exp(2 \ln \ln t)^{1/\nu} \le q_1 \exp(2 \ln Ct)^{1/\lambda}$$

which is not possible for $\nu < \lambda$, as $t \to \infty$.

Proposition 7. The spaces X_λ are not distinguishable by $\mathcal{D}_\varphi$.

Proof. By (1), an equivalent formulation of (4) is $\exists N, C : d_k \ge \varphi^{-N}(C b_k^{-N}), \quad \forall k$. Suppose X_λ has the property $\mathcal{D}_\varphi$ for some φ. Then $\exists N, C$ such that

$$\exp\exp(\ln k)^\lambda \le \varphi^N(C \exp(N (\ln k)^\lambda)).$$

Fix $\lambda \neq \nu$. For $j \in \mathbb{N}$ let $k = k(j)$ be such that $(\ln k)^\lambda \le (\ln j)^\nu < (\ln(k+1))^\lambda$. Without loss of generality we can assume $k \ge 2$. Then, taking into account the bound $\ln(k+1) < 2 \ln k$, we get

$$\begin{array}{rcl} (d_j^{(\nu)})^{-1} & = & \exp\exp(\ln j)^\nu \\ & \le & \exp\exp(\ln(k+1))^\lambda \\ & \le & \varphi^N(C\exp(N(\ln(k+1))^\lambda)) \\ & < & \varphi^N(C\exp(N 2^\lambda (\ln k)^\lambda)) \\ & < & \varphi^N(C\exp(N 2^\lambda (\ln j)^\nu)). \end{array}$$

Therefore, X_ν has the property $\mathcal{D}_\varphi$ with the same φ. In wiew of symmetry we conclude that the invariant $\mathcal{D}_\varphi$ is not able to distinguish the spaces $X_\lambda, \lambda > 1$.

VI. A Geometrical Necessary Condition for Isomorphism.

In this section we are going to use interpolation neigbourhoods together with the counting function β, and assume that for any $N > 1$, $d_k \le h_k^N$ asymptotically. Fix $\varepsilon = \varepsilon(p,q,r)$ such that $q < p\varepsilon + r(1-\varepsilon)$.

In our constructions the following interpolation lemma whose proof can be found in [21] will be used.

Lemma 1. If $T \in L(l^1(a^0), l^1(b^0)) \cap L(l^1(a^1), l^1(b^1))$ then $T \in L(l^1(a^\alpha), l^1(b^\alpha))$ where $0 \le \alpha \le 1$, $a_j^\alpha = (a_j^0)^{1-\alpha}(a_j^1)^\alpha$ and $b_j^\alpha = (b_j^0)^{1-\alpha}(b_j^1)^\alpha$. Further $\|T\|_\alpha \le \max\{\|T\|_0, \|T\|_1\}$.

In connection with the above lemma, it is customary to define for $X = K(b_{i,p})$

$$U_p^\varepsilon U_r^{1-\varepsilon} = \left\{ x = (x_i) \in X : \sum_{i=1}^{\infty} |x_i| b_{ip}^\varepsilon b_{ir}^{1-\varepsilon} \le 1 \right\}.$$

From the Lemma it follows that if $T(U_p) \subset V_{p'}$ and $T(U_r) \subset V_{r'}$, then $T(U_p^\varepsilon U_r^{1-\varepsilon}) \subset V_{p'}^\varepsilon V_{r'}^{1-\varepsilon}$.

VI.1 Counting function β_I.

In this part we choose the neighborhoods in (5) as $W_1 = U_p^\varepsilon U_r^{1-\varepsilon}$, $W_2 = tU_r$, $W_3 = W_4 = U_q$. Then we obtain a new function $\beta(U_p^\varepsilon U_r^{1-\varepsilon} \cap tU_r, U_q) := \beta_I$. By Proposition 5, $|I_1| \le \beta_I \le |I_2|$ where

$$I_1 := \left\{ i : \frac{b_{ip}^\varepsilon b_{ir}^{1-\varepsilon}}{b_{iq}} \le 1,\ \frac{b_{ir}}{b_{iq}} \le t \right\},\quad I_2 := \left\{ i : \frac{b_{ip}^\varepsilon b_{ir}^{1-\varepsilon}}{b_{iq}} \le 2,\ \frac{b_{ir}}{b_{iq}} \le 2t \right\}.$$

In our case we have $i = (n,k)$, $b_{ip} = a_{nkp} = n^p d_k^{-\min(n,p)} h_k^{-p}$. Then choosing $n = q$ we have

$$I_1 \supset \left\{ (q,k) : \frac{a_{qkp}^\varepsilon a_{qkr}^{1-\varepsilon}}{a_{qkq}} \le 1,\ \frac{a_{qkr}}{a_{qkq}} \le t \right\}. \tag{11}$$

The first inequality above is $q^{p\varepsilon+r(1-\varepsilon)-q}\, d_k^{\varepsilon(q-p)} \le h_k^{p\varepsilon+r(1-\varepsilon)-q}$. When k is large enough $q^{p\varepsilon+r(1-\varepsilon)-q} \le d_k^{-\frac{\varepsilon}{2}(q-p)}$. By our assumption on h_k and d_k for all large enough k we have $d_k \le h_k^{\frac{2(r-q)}{\varepsilon(q-p)}}$ and so $q^{p\varepsilon+r(1-\varepsilon)-q}\, d_k^{\varepsilon(q-p)} \le d_k^{\frac{\varepsilon}{2}(q-p)} \le h_k^{r-q} \le h_k^{p\varepsilon+r(1-\varepsilon)-q}$ which means that when $n = q$, the first inequality in (11) holds for all large enough k. The second inequality in (11) is equivalent to $(q/h_k)^{r-q} \le t$. Thus

$$I_1 \supset \left\{ k : \left(\frac{q}{h_k}\right)^{r-q} \le t \right\}.$$

Next we consider the first inequality in I_2. If $n > r$, it becomes

$$\frac{(n^p d_k^{-p} h_k^{-p})^\varepsilon\, (n^r d_k^{-r} h_k^{-r})^{1-\varepsilon}}{n^q d_k^{-q} h_k^{-q}} \le 2$$

that is $n^{p\varepsilon+r(1-\varepsilon)}d_k^q h_k^q \leq 2n^q(d_k h_k)^{p\varepsilon+r(1-\varepsilon)}$ which is impossible since $q < p\varepsilon + r(1-\varepsilon)$. Similarly if $n \leq p$, the first inequality in I_2 is impossible.

If $p < n \leq r$, then the second inequality in I_2 implies $\frac{1}{h_k^{r-q}} \leq 2t$. Thus

$$I_2 \subset \left\{(n,k) : p < n \leq r,\ \frac{1}{h_k^{r-q}} \leq 2t\right\} \Rightarrow |I_2| \leq (r-p)\left|\left\{k :\ \frac{1}{h_k^{r-q}} \leq 2t\right\}\right|,$$

and

$$\left|\left\{k : \left(\frac{q}{h_k}\right)^{r-q} \leq t\right\}\right| \leq \beta_I \leq r\left|\left\{k : \frac{1}{h_k^{r-q}} \leq 2t\right\}\right|.$$

Now assume $K(A_{(d,h)})$ is isomorphic to $K(A_{(\tilde{d},\tilde{h})})$ and they have systems of neighborhoods (U_p) and $(\tilde{U}_p)$ respectively. Assume also

$$d_k \leq h_k^N \quad \text{and} \quad \tilde{d}_k \leq \tilde{h}_k^N \tag{12}$$

asymptotically for all $N > 1$. We note that this is not really a restriction since for the other regular case (i.e. $d_k \geq h_k^{N_0}$ aymptotically for some $N_0 > 1$), it is known that $K(A_{(d,h)})$ is isomorphic to s, the space of rapidly decreasing sequences. Then by isomorphism

$$\forall p_1\ \exists p\ \forall q\ \exists q_1\ \forall r_1\ \exists r\ \exists C :\ \beta(U_p^{\varepsilon}U_r^{1-\varepsilon} \cap tU_r, U_q) \leq \beta(C(\tilde{U}_{p_1}^{\varepsilon}\tilde{U}_{r_1}^{1-\varepsilon} \cap t\tilde{U}_{r_1}), \tilde{U}_{q_1}).$$

Fixing p_1 and p, we get

$$\forall q\ \exists q_1\ \forall r_1\ \exists r\ \exists C : \left|\left\{j : \left(\frac{q}{h_j}\right)^{r-q} \leq t\right\}\right| \leq r_1\left|\left\{j : \frac{1}{\tilde{h}_j^{r_1-q_1}} \leq Ct\right\}\right|.$$

Fix k and choose $t = \left(\frac{q}{h_k}\right)^{r-q}$. Since the sequence (h_j) is monotonically decreasing, the set on the left hand side has exactly k elements. Thus the set $\{j :\ 1/\tilde{h}_j^{r_1-q_1} \leq Ct\}$ has at least $i(k/r_1)$ elements (where for a real number α, $i(\alpha)$ denotes the smallest integer which is greater than or equal to α.) Since $(\tilde{h}_i)$ is monotonic, the integer $j = i(k/r_1)$ belongs to the set $\{j :\ 1/\tilde{h}_j^{r_1-q_1} \leq Ct\}$. Now we use the notation

$$h_\alpha = \begin{cases} h_\alpha & \text{if } \alpha \in \mathbb{N} \\ h_{i(\alpha)} & \text{if } \alpha \notin \mathbb{N} \end{cases}$$

With this notation $\frac{1}{\tilde{h}_{\frac{k}{r_1}}^{r_1-q_1}} \leq C\left(\frac{q}{h_k}\right)^{r-q}$. This and the symmetrical inequality together imply the following theorem.

Theorem 2. Let $K(A_{(d,h)})$ be isomorphic to $K(A_{(\tilde{d},\tilde{h})})$ and assume that (12) holds. Then there is an $M > 1$ such that

$$h_k^M \leq M\tilde{h}_{\frac{k}{M}} \quad \text{and} \quad \tilde{h}_k^M \leq Mh_{\frac{k}{M}}.$$

VI.2 Counting function β_{II}.

This time we take $W_1 = W_4 = U_q, W_2 = U_p^\varepsilon U_r^{1-\varepsilon}, W_3 = \tau U_p$ in (5). So, we obtain

$$\beta(U_q \cap U_p^\varepsilon U_r^{1-\varepsilon}, \overline{conv}(\tau U_p \cup U_q)) := \beta_{II}.$$

We proceed as in the previous subsection. We define

$$\begin{aligned} J_1 &= \left\{ i : \frac{b_{iq}}{b_{ip}} \leq \frac{1}{\tau}, \ \frac{b_{iq}}{b_{iq}} \leq 1, \ \frac{b_{ip}^\varepsilon b_{ir}^{1-\varepsilon}}{b_{ip}} \leq \frac{1}{\tau}, \ \frac{b_{ip}^\varepsilon b_{ir}^{1-\varepsilon}}{b_{iq}} \leq 1 \right\}, \\ J_2 &= \left\{ i : \frac{b_{iq}}{b_{ip}} \leq \frac{2}{\tau}, \ \frac{b_{iq}}{b_{iq}} \leq 2, \ \frac{b_{ip}^\varepsilon b_{ir}^{1-\varepsilon}}{b_{iq}} \leq \frac{2}{\tau}, \ \frac{b_{ip}^\varepsilon b_{ir}^{1-\varepsilon}}{b_{iq}} \leq 2 \right\}. \end{aligned}$$

Then by Propositions 4 and 5, $|J_1| \leq \beta_{II} \leq |J_2|$. In J_1, the second inequality is trivial and third inequality follows from the first and the fourth, and in J_2, the second inequality is trivial and dropping the third inequality may only enlarge the set. For these sets we proceed as in the previous subsections and obtain (with q fixed)

$$\begin{aligned} J_1 &\supset \left\{ (q,k) : \frac{a_{qkq}}{a_{qkp}} \leq \frac{1}{\tau} \right\} = \left\{ (q,k) : \left(\frac{q}{d_k h_k} \right)^{q-p} \leq \frac{1}{\tau} \right\} \\ J_2 &\subset \left\{ (n,k) : p < n \leq r, \ \frac{1}{d_k h_k} \leq \frac{2}{\tau} \right\}. \end{aligned}$$

Thus $|\{k : \left(\frac{q}{d_k h_k}\right)^{q-p} \leq \frac{1}{\tau}\}| \leq \beta_{II} \leq (r-p)|\{k : \frac{1}{d_k h_k} \leq \frac{2}{\tau}\}|$. Arguing exactly as in Theorem 2, we obtain

Theorem 3. Let $K(A_{(d,h)})$ be isomorphic to $K(A_{(\tilde{d},\tilde{h})})$ and assume (12) holds. Then there is an $M > 1$ such that

$$(d_k h_k)^M \leq M \tilde{d}_{\frac{k}{M}} \tilde{h}_{\frac{k}{M}} \ \text{ and } \ (\tilde{d}_k \tilde{h}_k)^M \leq M d_{\frac{k}{M}} h_{\frac{k}{M}}.$$

The following corollary follows from Theorems 2 and 3 immediately.

Corollary. Assume (12) holds and $K(A_{(d,h)})$ is isomorphic to $K(A_{(\tilde{d},\tilde{h})})$. Then there is an $M > 1$ such that

$$d_k^M \leq M \tilde{d}_{\frac{k}{M}}, \ \tilde{d}_k^M \leq M d_{\frac{k}{M}}, \ h_k^M \leq M \tilde{h}_{\frac{k}{M}}, \ \tilde{h}_k^M \leq M h_{\frac{k}{M}}. \tag{13}$$

VII. Example of non-isomorphic spaces $K(A_{(d,h)})$ not distinguishable by $\beta(\tau U_p \cap t U_r, U_q)$.

We fix $\lambda > 1$ and consider the space $X_\lambda = K(A_{(d,h)})$ with $d_k = \exp(-k^\lambda), \ h_k = k^{-2}$.

Proposition 8. If $\lambda \neq \nu$, then the spaces X_λ, X_ν are not isomorphic.

Proof. Suppose contrary to our claim that they are isomorphic for some $\nu > \lambda > 1$. Then (13) implies that $\exists M$ such that $e^{(\frac{k}{M})^\nu} \leq M e^{Mk^\lambda}$ which is not possible for any M when $\nu > \lambda, k \to \infty$.

It is easy to see that the spaces X_λ are not distinguishable by both $\beta(\tau U_p \cap tU_r, U_q)$ and $\mathcal{D}_\varphi$.

Recall that in [7] an example of a continuum of pairwise non-isomorphic spaces $\mathcal{E}_0(K_\lambda)$ was given using the invariant $\mathcal{D}_\varphi$. In [7] it also was shown that for $h_k \sim \frac{1}{k^M}$ if $\mathcal{E}_0(K_d)$ is isomorphic to $\mathcal{E}_0(K_\delta)$, then for some constant $N > 1$, $d^N_{k^N} \leq C\delta_k$ and $\delta^N_{k^N} \leq Cd_k$. In the present paper, this necessary geometric condition has been improved, namely the subscript k^N has been replaced by the linear one Mk.

Thus we have the sequence $\mathcal{D}_\varphi$, $\beta(\tau U_p \cup tU_r, U_q)$, β_I and β_{II} of linear topological invariants of increasing complexity. Using the invariant from this sequence we can present a continuum of pairwise non-isomorphic spaces of type $\mathcal{E}(K)$ or $\mathcal{E}_0(K)$ nondistinguishable by previous invariants. These examples show that the topological structure of the spaces of Whitney functions is rather complicated even for our model case: K is a sequence of closed intervals tending to a point. Complete classification problem is still open.

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